S. KOVALEVSKY: A MATHEMATICAL LESSON

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Sofya Kovalevsky was a noted writer whose works include both fiction and nonfiction. She was also a political activist and a public advocate of feminism. In addition, she was a brilliant mathematician who made significant contributions despite the enormous educational and political obstacles that she had to overcome. Somehow her many achievements have been forgotten. In those few instances where her work has not been lost it has been denigrated by such studies as Felix Klein's history of nineteenth-century mathematics. Klein dismisses Kovalevsky's work in the following manner: "Her works are done in the style of Weierstrass and so one doesn't know how much of her own ideas are in them."[1] He finds something wrong with all her research and credits her with only one positive accomplishment, drawing Weierstrass out of his shell through their correspondence. It is time to set this record straight and to let the facts speak for themselves.

Sofya Krukovsky, known affectionately as Sonya, was born in Moscow in 1850. Her father, a Russian army officer, retired in 1858 and moved the family—Sofya, her older sister, Anyuta, and her younger brother, Fedya—to Palibino, an estate near the Lithuanian border.

After settling at Palibino, the household discovered that they had not brought a sufficient amount of wallpaper with them. Rather than travel a great distance to obtain new wallpaper, they decided to use old newspapers on the wall. Since only the nursery required the paper, this was deemed an adequate solution. However, while searching the attic for newspaper, they discovered paper of a better quality. On it were the lecture notes from a calculus course taken by General Krukovsky. This is how the nursery walls came to be covered with the calculus notes that, in her later years, Sofya claimed to have studied. Sofya often repeated this anecdote and enjoyed reporting how her calculus teacher exclaimed: "You have understood them as though you knew them in advance."[2]

Kovalevsky claimed that her interest in mathematics was aroused by her Uncle Peter, who would discuss numerous abstractions and mathematical concepts with her. When the family tutor, Joseph Malevich, read of this in Sofya's autobiographical work, *Memories of Childhood*, he was incensed. He wrote a long essay in a Russian newspaper explaining why *he* should receive credit for Kovalevsky's mathematical development. In response to this criticism Kovalevsky wrote the following tribute in "An Autobiographical Sketch": "It is to Joseph Malevich that I am indebted for my first systematic study of mathematics. It happened so long ago that I no longer remember his lessons at all.... It was arithmetic that Malevich taught best...I have to confess that arithmetic held little interest for me."[3]

Kovalevsky[4] studied mathematics against her father's wishes. When she was thirteen, she smuggled an algebra text into her room and studied it. When she was fourteen she taught herself trigonometry in order to study a physics book written by her neighbor Professor Tyrtov—trigonometry was necessary for the optics section, and the young Sofya taught herself without tutor or text. By constructing a chord on a circle, she was able to explain the sine function and to develop the other trigonometric formulas. When Professor Tyrtov saw her work, he was struck by its similarity to the actual mathematical development. Calling her a new Pascal, Tyrtov pleaded with the General to permit Sofya to study mathematics. After a year of exhortation, General Krukovsky relented and allowed Sofya to go to Petersburg to study calculus and other subjects.

After completing her studies in 1867, Sofya wanted to continue her education, but the Russian university system was closed to women. The only option for study was to go to Switzerland, but General Krukovsky would not allow his daughters to go abroad.

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Sofya's older sister, Anyuta, felt imprisoned at Palibino and sought a way out. She found it through the radical politics of the times. This was a period of political ferment in Russia. The nihilists, feminists, and radicalists were all active, and their ideas were brought to Palibino by the local priest's son on his vacation from school in Petersburg. While they scandalized the neighborhood, these ideas had great influence on Anyuta, who in turn influenced Sofya. Anyuta joined a radical group that advocated higher education for women and promoted the concept of the "fictitious husband" to enable women to obtain more freedom. A married woman did not need her father's signature for a passport, and so a fictitious husband would enable Anyuta to travel abroad for her education. Anyuta and her friend Zhanna found a 26-year-old university student, Vladimir Kovalevsky, who agreed to marry one of them. Unfortunately, for Anyuta, she brought Sofya to one of their meetings. Vladimir became infatuated with Sofya and insisted on marrying her. After several secret meetings and much intrigue, General Krukovsky consented, and Vladimir and Sofya were married in September 1868.

Following their marriage, the Kovalevskys left for Petersburg to study, and to search for a husband for Anyuta. With little effort Sofya had won the freedom to pursue her education, the freedom and independence that Anyuta had been fighting so hard for. Sofya's feelings of guilt about this can be seen in her letters to Anyuta, who was still confined at Palibino. She wrote: "At times a strange anguish comes over me and I feel ashamed that everything is coming to me so easily and without any struggle."[5]

In Petersburg, Sofya received permission from the instructors to attend classes *unofficially*. She wrote to her sister: "... Lectures begin tomorrow and so my real life begins at 9 A.M. ... [Vladimir] and friends will solemnly escort me by way of the backstairs so that there is hope of hiding from the administration and from curious stares."[6] It was at Petersburg that Sofya decided to concentrate on mathematics. In a letter to Anyuta she said: "I have become convinced that one cannot learn everything and one life is barely sufficient to accomplish what I can in my chosen field."[7]

The Kovalevskys and Anyuta, who was still unmarried but chaperoned by the young couple, left for Europe in 1869. Sofya intended to study mathematics; and Vladimir, geology. Anyuta planned to pursue her revolutionary activities. Sofya and Vladimir settled in Heidelberg, but Sofya was not permitted to matriculate at the university. She appealed to both the faculty and the administration. A special committee was formed, and it was decided that each individual professor could choose whether to permit Sofya Kovalevsky to attend his lectures unofficially.

Kovalevsky was now able to attend lectures, and her outstanding mathematical ability became the talk of Heidelberg. As a firm believer in education for women, she used her reputation to assist other Russian women in their efforts to attend the university. One of these women was her friend Yulya Lermontov, who later became the first female chemist in Russia. For many years Bunsen had described Sofya as "a dangerous woman"[8] because, according to him, Sofya had tricked him into permitting Yulya to use the previously all-male chemistry labs. In 1874, Karl Weierstrass asked Sofya for confirmation of the story because "he [Bunsen] writes fiction even if he doesn't publish it."[9]

Sofya, Vladimir, and Yulya lived and studied together in Heidelberg until the fall of 1869. When Anyuta arrived for a visit, she was quite surprised to find Sofya still living with her "fictitious husband" and proceeded to evict Vladimir from the apartment. A short time later, Vladimir left Heidelberg to study at Jena.

As her mathematics progressed, Kovalevsky felt the need to study with Karl Weierstrass, the most noted mathematician of the time, at the University of Berlin. She traveled to Berlin for the start of the fall semester of 1870, only to find the university closed to women. Sofya wrote: "The capital of Prussia proved to be backward. Despite all my pleadings and efforts I had no success in obtaining permission to attend the University of Berlin."[10]

Determined to study mathematics, Kovalevsky personally presented herself to Weierstrass as an aspiring student. On the basis of recommendations from Koenigsberger at Heidelberg, Weierstrass was willing to see her. He assigned her a set of problems on hyperelliptic functions that he had just given to his class. Weierstrass was so impressed with the ability she demonstrated in her solutions that he personally requested the university to allow her to attend his classes unofficially. However, the university was intransigent in its decision.

Not wanting to waste this mathematical talent, Weierstrass offered to tutor Kovalevsky privately. The lessons, begun in the fall of 1870, lasted for four years. The working relationship lasted a lifetime. When Weierstrass made his offer, he had no idea that Kovalevsky would become the closest of his disciples and remain so until her death.

The lessons began with twice-weekly meetings, Sundays at Weierstrass's home and weekdays at Kovalevsky's. The mathematics lessons are partially documented in a series of forty-one letters from Weierstrass to Kovalevsky, spanning the period of March 1871 to August 1874. They show that the emphasis was on Weierstrass's favorite topic: Abelian functions. Kovalevsky's responses are unavailable, because Weierstrass had burned them on hearing of her death. However, some idea of the importance of this correspondence to Kovalevsky can be found in her writings: "These studies had the deepest possible influence on my entire career in mathematics. They determined finally and irrevocably the direction I was to follow in my later scientific work: all my work has been done precisely in the spirit of Weierstrass."[11]

These letters also show the development of a close personal relationship between Kovalevsky and Weierstrass. They give a glimpse of the growing affection that Weierstrass felt for his pupil. Weierstrass was unaware of Kovalevsky's marriage arrangement and did not understand Vladimir's appearances. It was not until two years later (October 1872) that Weierstrass learned the truth. The series of letters at this time indicate that the topic of a relationship between Kovalevsky and Weierstrass was raised and the truth about her marital status was made known. Both were upset at the scene, and Weierstrass made several attempts to reassure his pupil that he would thereafter only discuss science. For a short period the correspondence remained strictly mathematical, but it did not stay that way. Weierstrass was not able to mask his concern and affection for Kovalevsky.

In October of 1872, Weierstrass had suggested several possible topics for Kovalevsky's dissertation. By 1874, she had completed three original works, any one of which Weierstrass felt would be acceptable. Now he needed to find a university that would award Kovalevsky a degree. In July 1874, the University of Göttingen awarded Sofya Kovalevsky a Ph.D., in absentia, summa cum laude, without either orals or defense. Needless to say, this was an unprecedented event.

The three papers presented for the degree were:

1. "On the Theory of Partial Differential Equations"
2. "On the Reduction of a Certain Class of Abelian Integrals of the Third Rank to Elliptic Integrals"
3. "Supplementary Remarks and Observations on Laplace's Research on the Form of Saturn's Rings"

The first of the papers, on partial differential equations, was published in Crelle's journal in 1875. This was considered a great honor, especially for a novice mathematician, since Crelle's journal was considered the most serious mathematical publication in Germany.

In the first paper, Kovalevsky had generalized a problem that had been posed by Cauchy. Cauchy had examined an existence theorem for partial differential equations, and Kovalevsky generalized Cauchy's results to systems of order r containing time derivatives of order r. The mathematician H. Poincaré said that "Kovalevsky significantly simplified the proof and gave the theorem its definitive form."[12] Today, this theorem on the existence and uniqueness of solutions of partial differential equations is often, but not always, known as the Cauchy-Kovalevsky Theorem. While studying the partial differential equation problem, Kovalevsky examined the heat equation. Some of her results were helpful to Weierstrass and he wrote: "So you see, dear Sonya, that your observation (which seemed so simple to you) on the distinctive property of partial differential equations...was for me the starting point for interesting and very elucidating researches."[13]

Shortly after Sofya received her degree from Göttingen, her friend Yulya was awarded a degree in chemistry from Göttingen. Vladimir had received his degree in paleontology two years earlier.

Although she had earned her Ph.D. and had written a highly acclaimed paper, Sofya Kovalevsky was unable to get a job. Even the efforts of her mentor, Weierstrass, were fruitless. So, in 1875, together with Vladimir, Anyuta, and Anyuta's husband, Sofya returned to the family home in Russia. Weierstrass encouraged her to relax at home and "enjoy the pleasures of big city life. I know you won't give up your scientific work."[14] However, Kovalevsky did not actively pursue her work that winter. She felt guilty and wrote: "I worked far less zealously than I had done in Germany and, indeed, the situation was far less propitious for scholarly work ... The only thing that still gave me scholarly support was the exchange of letters and ideas with my beloved teacher Weierstrass."[15] As important as this exchange may have been, the correspondence with Weierstrass stopped in 1875. It was resumed for a short time in 1878 but did not become regular until 1880. Weierstrass was very hurt by this neglect. However, it must be acknowledged that Kovalevsky was never a good correspondent, even while in Germany, so that it is not surprising to find this lapse in writing after her return to Russia.

In Russia, Kovalevsky again tried to find a job. The only position in mathematics available to a woman was teaching arithmetic in the lower grades of a girls' school, and since Sofya admitted that she was "unfortunately weak in the multiplication table"[16] she could not seriously consider such a position. Therefore, Kovalevsky turned to other intellectual pursuits. She wrote fiction, theater reviews, and popular-science reports for a newspaper. She was instrumental in the organization of the Bestuzhev School for Women, but because of what were considered her radical views she was not permitted to teach there.

During this period in Russia, General Krukovsky died and left Sofya a small inheritance. Vladimir invested this money in business enterprises that eventually went bankrupt.

By 1878, Sofya had become bored with her activities and wanted to return to mathematics. She wrote to Weierstrass for advice. Weierstrass was excited by this letter, the first he had received from his pupil in three years. However, Kovalevsky's return to mathematics was delayed by the birth of a daughter, Sofya Vladimirovna, in October of 1878.

On their return to Russia the Kovalevskys had assumed the obligations of a real marriage. This was done partly as an obligation to Sofya's parents and partly because of their new politics. It was their feeling to end lying relationships of all kinds, and so the marriage was finally consummated.

Kovalevsky's return to mathematics was encouraged by a scientific conference held in Petersburg in 1880. The Russian mathematician Chebyshev invited Kovalevsky to present a paper at this conference. She found her unpublished dissertation on Abelian Integrals, translated it from German to Russian in one night, and presented it to the conference. Although it had lain untouched for six years, it was well received by the mathematicians.

Following her presentation, the Swedish mathematician Gösta Mittag-Leffler, who had met Kovalevsky earlier while she was a student in Germany, offered to find her a position in his country. Kovalevsky was very appreciative of this offer and wrote, in 1881, to Mittag-Leffler: "[If I can teach] I may in this way open the universities to women, which have hitherto only been open by special favor, a favor which can be denied at any moment."[17]

Kovalevsky's desire for a position was spurred not only by her feminism but also by her need to do mathematics. She wished "at the same time, to be able to live for my work, surrounded by those who are occupied with the same questions."[18]

While waiting for word from Sweden, Kovalevsky looked in Berlin for research work. While Vladimir was on a business trip, Sofya secretly visited Weierstrass. When she decided that she would return to Berlin to pursue research, Vladimir was quite angry with the decision and their marriage ended. For the next two years, Kovalevsky lived a student's life in Berlin, and the care of her daughter was shared by Sofya's friend Yulya and her brother-in-law Alexander.

While Sofya Kovalevsky was busy conducting her research on the refraction of light in crystals, Vladimir Kovalevsky again managed to get himself into financial difficulties. There was a stock scandal, and Vladimir, faced with ruin, committed suicide in the spring of 1883. Sofya took solace in her mathematics and hoped to find a badly needed job in Stockholm.

Mittag-Leffler had recently been appointed the head of the mathematics department at the newly founded University of Stockholm and was able to offer Sofya a position there. However, the job had conditions. In order to prove her competence, Kovalevsky was to teach for a probationary year, with no pay and no official university affiliation. Kovalevsky agreed to this because she had no other options.

In the fall of 1883, Sofya Kovalevsky arrived in Stockholm to become a lecturer at the University of Stockholm. Her reception was mixed. Although hailed by many, others agreed with A. Strindberg, who wrote in the local paper, "A female professor is a pernicious and unpleasant phenomenon—even, one might say, a monstrosity."[19]

In the spring of 1884, Kovalevsky lectured in German on partial differential equations. These lectures were well received, and Mittag-Leffler was able to obtain the funds for her appointment as a Professor of Higher Analysis in July 1884. Word of the appointment was sent to Sofya in Moscow, where she was spending the summer with her daughter. However, Sofya did not bring her daughter to Stockholm in the fall because she was still unsure of her position. She was publicly criticized for her child-care arrangements but chose to ignore this. She didn't bring her daughter to Sweden until 1885, when the child was eight years old.

In addition to joining the Stockholm faculty in 1884, Kovalevsky became an editor for *Acta Mathematica* and published her first paper on crystals. In 1885, she received a second appointment to the Chair of Mechanics. She also published a second paper on the propagation of light in crystals but was embarrassed when Volterra found a serious error in her work. She had used a multi-valued function as if it were a single-valued one. Since this research was performed after a three-year hiatus in her mathematical career, Kovalevsky felt that her teacher, Weierstrass, should have caught the error prior to publication. Weierstrass, quite distraught, blamed illness and overwork. (It might be added that Weierstrass was 70 years old at the time.)

While in Stockholm, Sofya lived for a time in Mittag-Leffler's home, where she met and developed a friendship with his sister, Anna Leffler. Anna Leffler, a well-known advocate of women's rights and a writer, encouraged Sofya's literary leanings. In 1887 they collaborated on a play entitled *The Struggle for Happiness*. It was based on an idea that had occurred to Sofya while she sat at the bedside of her dying sister.

After Anyuta died in the fall of 1887, Sofya felt lonely and despondent. The sisters had been close, and Sofya felt the loss deeply. However, at this time two events occurred that helped to assuage her grief. Both the announcement of a new competition for the Prix Bordin and the arrival in Stockholm of a Russian lawyer named Maxim Kovalevsky were to have profound effects on the life of Sofya Kovalevsky.

Early in 1888 the French Academy of Science announced a new competition for the Prix Bordin. Papers on the theory of the rotation of a solid body would be considered for the prize competition at the end of the summer. Gösta Mittag-Leffler encouraged Sofya to work on a paper for the competition.

While Sofya was engaged in her research on the paper, Maxim Kovalevsky arrived to give a series of lectures at Stockholm University. He had been dismissed from Moscow University for criticizing Russian constitutional law. Aside from politics, Sofya and Maxim had many interests in common, and their attraction resulted in a scandalous affair. Eventually Maxim proposed marriage, on the condition that Sofya give up her research. Even if she had wished to give up her mathematics, Sofya was too far into her work for the prize competition to stop. In order to free Sofya to do her work, Mittag-Leffler invited Maxim to his summer home in Uppsala. This was a wise move, for, as Sofya stated, "If burly Maxim had stayed longer, I do not know how I should have got on with my work."[20]

With Maxim gone, Sofya was able to finish her work and the paper was submitted on time. Of the fifteen papers, which were submitted anonymously, one was considered so outstanding that the award was increased from 3,000 francs to 5,000 francs. The Prix Bordin was awarded to Sofya Kovalevsky in December of 1888. Sofya attended the awards ceremony with Maxim. Special recognition was given to her work. In his congratulatory speech, the President of the Academy of

Sciences said: "Our co-members have found that her work bears witness not only to profound and broad knowledge, but to a mind of great inventiveness."[21]

Prior to Sofya Kovalevsky's work the only solutions to the motion of a rigid body about a fixed point had been developed for the two cases where the body is symmetric. In the first case, developed by Euler, there are no external forces, and the center of mass is fixed within the body. This is the case that describes the motion of the earth. In the second case, derived by Lagrange, the fixed point and the center of gravity both lie on the axis of symmetry of the body. This case describes the motion of the top. Sofya Kovalevsky developed the first of the solvable special cases for an unsymmetrical top. In this case the center of mass is no longer on an axis in the body. She solved the problem by constructing coordinates explicitly as ultra-elliptic functions of time. Kovalevsky continued this work in two more papers on a rigid body motion. These both received awards from the Swedish Academy of Sciences in 1889. Her later works on the subject have been lost.

Kovalevsky's professorship in Sweden was due to expire in 1889. Desirous of returning to her native country, she inquired about a position in Russia. Her request was flatly denied. Russian mathematicians were indignant at this slight of Kovalevsky and decided to honor her. It was suggested that an honorary membership in the Russian Academy of Sciences would provide that recognition. However, in order to do that, the charter had to be amended to allow for female membership. In November 1889, Chebyshev sent the following telegram to Kovalevsky: "Our Academy of Sciences has just now elected you a corresponding member, having just permitted this innovation for which there has been no precedent until now. I am very happy to see this fulfillment of one of my most impassioned and justified desires."[22]

Kovalevsky sought positions throughout Europe but was again unsuccessful. She therefore had to accept the renewal of the professorship in Stockholm.

While working in Stockholm, Kovalevsky regularly commuted to France, where she visited Maxim at his villa. During these visits Maxim encouraged her literary interests. She wrote *Memories of Childhood* in Russian. It was translated into Swedish and published in 1889. In order not to shock Swedish society with its personal revelations, it was released as a novel entitled *The Raevsky Sisters*. The original was published in Russian in 1890. Kovalevsky's novel *A Nihilist Girl* was written in Swedish in 1890. A Russian version had been started, but Sofya's sudden death left it unfinished. Maxim edited the two versions and was responsible for the posthumous publication of the novel. This book was highly praised by critics in Russia and Scandinavia.

It was Kovalevsky's frequent trips to France to visit Maxim that eventually caused her death. On returning to Stockholm from a visit, early in 1891, she fell ill. The illness was misdiagnosed, and by the time it was finally found to be pneumonia it was out of control. On February 10, 1891, Sofya Kovalevsky died. Although there was widespread mourning and eulogies were given around the world, the Russian Minister of the Interior, I. N. Durnovo, was concerned that too much attention was being paid to "a woman who was, in the last analysis, a Nihilist."[23]

The mathematical world was more generous in its praise. Mittag-Leffler gave the official eulogy for the University of Stockholm. Speaking of her as a teacher he said: "We know with what inspiring zeal she explained [her] ideas... and how willingly she gave the riches of her knowledge."[24] In his eulogy, Kronecker, of the University of Berlin, spoke of Kovalevsky as "one of the rarest investigators."[25] Karl Weierstrass, who felt her loss most deeply, having burned all of her letters, said " 'People die, ideas endure': it would be enough for the eminent figure of Sofya to pass into posterity on the lone virtue of her mathematical and literary work."[26]

In her short lifetime, Sofya Kovalevsky left a notable collection of political, literary, and mathematical works. Her contributions, completed in spite of many obstacles, certainly warrant her a place in our intellectual and mathematical history.

During her mathematical career Sofya Kovalevsky published ten papers in mathematics and mathematical physics. Three of these papers,[27, 28, 29] were written during her student days under Weierstrass (1870–1874). The articles on the refraction of light[30, 31, 32] were written years later in

Berlin (1881–1883) after her return to her mathematical researches. The remaining research on rigid body motion and Bruns's Theorem[33–36] was completed during her tenure at the University of Stockholm (1883–1891). The only complete collection of Kovalevsky's works is published in Russian.[37] Portions of her work in partial differential equations and rigid body motion appear in English, and since these are Kovalevsky's most important works they will be discussed in some detail.

The proof of the first general existence theorem in partial differential equations was presented by Cauchy in 1842, in his publications on integrating differential equations with initial conditions. Here he showed the existence of analytic solutions for ordinary differential equations and certain linear partial differential equations.

The Cauchy problem for the ordinary differential equation $du/dt = f(t,u)$, with the initial conditions $u = u_0$ and $t = t_0$ states:

> If $f(t,u)$ is an analytic function in a neighborhood of the point (t,u) there exists a unique solution of $du/dt = f(t,u)$ with the given initial conditions.

For systems of first-order partial differential equations of the form

$$\frac{\partial \mu_i}{\partial t} = F_i\left(t, x_1, \ldots, x_n; u_1, \ldots, u_n; \frac{\partial u_1}{\partial x_1}, \ldots, \frac{\partial u_m}{\partial x_m}\right)$$

with initial conditions when $t = 0$, that is $u_i(0, x_1, \ldots, x_n) = w_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, m$, the Cauchy problem is to find a solution $u(x, t)$ that satisfies the initial conditions.[38]

To solve this problem Cauchy assumed that F_i and w_i were analytic. He obtained a locally convergent power series solution by using his "method of majorants." The original function F_i is replaced by a simple analytic function whose power series expansion coefficients are nonnegative and greater than or equal to the absolute value of the corresponding coefficients for F_i. The derived system is then explicitly integrated to give a solution which is the majorant for the solution to the original with $t = 0$.

In her thesis Kovalevsky generalized Cauchy's results to systems of an order r containing time derivatives, $\partial^r u_i/\partial t^r$, of order r. It is this generalization that is known as the Cauchy-Kovalevsky Theorem.

Kovalevsky used majorants of the form

$$\frac{M}{1 - \left[(t_1 + t_2 + \cdots + t_r)/\rho\right]},$$

where ρ and M are constants.

For the higher order system

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = F_i\left(t, x_1, \ldots, x_n; u_n; u_1, \ldots, u_m; \frac{\partial^k u_j}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}\right)$$

with $i, j = 1, 2, \ldots, m$; $k_0 + k_1 + \cdots + k_n = k \leqslant n_j$, $k_0 < n_j$, and initial values given at some point $t = t_0$ for the u_i and their first $n_i - 1$ derivatives with respect to t, Kovalevsky proved the following.

If all the functions F_i are analytic in a neighborhood of the point $(t, x_1, \ldots, x_n; \ldots w_{j,k_0,k_1,\ldots,k_n})$ and all the functions $w_j^{(k)}$ are analytic in a neighborhood of the point $(x_1, \ldots, x_n)$, then Cauchy's problem has an analytic solution in a certain neighborhood of the point $(t, x_1, x_2, \ldots, x_n)$, and it is the unique solution in the class of analytic functions. (Note that

$$w_{j,k_0,k_1,\ldots,k_n} = \partial^{k-k_0} u_j / \partial x_1^{k_1} \ldots \partial x_n^{k_n}.)$$[39]

The simplest form of this theorem states that any equation of the form $\partial u/\partial t = f(t, x, u, \partial u/\partial x)$,where f is analytic in its arguments for values near $(t_0, x_0, u_0, \partial u_0/\partial x_0)$, possesses one and only one solution $u(t, x)$ which is analytic near (t_0, x_0).

The Cauchy-Kovalevsky Theorem has significant limitations. It is restricted to analytic functions, and convergence may fail in a region of interest. Also the computation of the coefficients of the series may be too tedious to be practical. However, it is important in that it shows that within a class of analytic solutions of analytic equations the number of arbitrary functions needed for a general solution is the same as the order of the equation and these arbitrary functions involve one less independent variable than the number occurring in the equation.

The work for which Kovalevsky was best known in her time was her research on the motion of a rigid body about a fixed point. The equations of motion of a rigid body moving about a fixed point were derived by Euler in 1750. They are as follows:

$$\frac{d\gamma}{dt} = r\gamma' - q\gamma'' \qquad \frac{Adp}{dt} + (C - B)qr = Mg(y_0\gamma'' - z_0\gamma'),$$
$$\frac{d\gamma'}{dt} = p\gamma'' - r\gamma \qquad \frac{Bdq}{dt} + (A - C)rp = Mg(z_0\gamma - x_0\gamma''),$$
$$\frac{d\gamma''}{dt} = q\gamma - p\gamma' \qquad \frac{Cdr}{dt} + (B - A)pq = Mg(x_0\gamma' - y_0\gamma).$$

Here A, B, C are the principal axes of the ellipsoid of the body relative to the fixed point; M is the mass; g is the acceleration due to gravity. $\gamma, \gamma', \gamma''$ are the direction cosines of the angles which the three axes make at each moment; their direction is the same as the force of the rigid body; p, q, r are the components of the angular velocity along the principal axes; x_0, y_0, z_0 are the coordinates of the center of gravity of the body considered in a system of coordinates of which the origin is the fixed point and whose direction coincides with that of the principal axes of the ellipsoid of inertia.

The problem to be solved was the integration of this system of equations so that the position of the moving body at any time could be obtained. Before 1888, the integration had been completed for only two cases. The first was for the condition $x_0 = y_0 = z_0 = 0$. This case, studied by Euler and Poisson, is the one where the center of gravity of the moving body coincides with the fixed point. This is the motion of a force-free symmetric body. There are no external forces acting on the body and the motion is about a fixed point within the body, the center of mass. If the fixed point is the center of gravity, then gravity does not influence the motion. The axes of rotation are thus fixed in the body. An example of this force-free motion is the free rotation of the earth. In the case of the earth's free rotation, the axis of rotation does not coincide with the axis of symmetry. It is very slightly tilted, although it passes through the center of mass of the earth. What then happens is that the instantaneous axis precesses slowly about the axis of symmetry.

In the second case, studied by Lagrange, $A = B$, $x_0 = y_0 = 0$. Here the fixed point and the center of gravity lie on the same axis. When this axis is the axis of symmetry, the motion is that of the spinning top. "A top is defined to be a material body which is symmetrical about an axis and terminates in a sharp point ... at one of the axis."[40] This top spins about a fixed point that is not the center of gravity, but the center of gravity and the fixed point both lie on the axis of symmetry of the top. The weight of the top gives rise to a moment of force as it spins about the fixed point on the plane.

In both of these cases the rigid body was symmetrical. Sofya Kovalevsky developed the first of the soluble special cases for the heavy or unsymmetrical top. In this case, the center of mass no longer lies on the axis of the body. Instead, it is in the equatorial plane (the plane perpendicular to the axis) and passes through the fixed point. In addition, two of the principal moments of inertia are equal and double the third. The center of gravity is in the plane of the equal moments of inertia.

Euler's equations containing six unknown functions has the following first integrals:

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma + y_0\gamma' + z_0\gamma'') = C_1$$
$$Ap\gamma + Bq\gamma' + Cr\gamma'' = C_2$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = C_3 = 1.$$

It is sufficient to find one more integral to obtain a complete solution in quadratures. This occurs because the time variable appears only in the form dt and can be eliminated, leaving only five equations.

Kovalevsky derived the fourth integral for the case $A = B = 2C$, $z_0 = 0$,[41] and showed that the only conditions necessary for a given series to be a solution to Euler's system are that the constants A, B, C, x, y, z satisfy one of four conditions:

(1) $A = B = C$,
(2) $x_0 = y_0 = z_0 = 0$ (Euler's case),
(3) $A = B$, $x_0 = y_0 = 0$ (Lagrange's case),
(4) $A = B = 2C$, $z_0 = 0$ (Kovalevsky's case).[42]

Kovalevsky obtained a solution for her case by rotating the coordinate axes in the xy plane and choosing a unit of length so that $y_0 = 0$ and $C = 1$. With $c_0 = Mgx_0$ the Euler equations become:

$$2\frac{dp}{dt} = qr \qquad \frac{d\gamma}{dt} = r\gamma' - q\gamma''$$
$$2\frac{dg}{dt} = -pr - c_0\gamma'' \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma$$
$$2\frac{dr}{dt} = c_0\gamma' \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma'.$$

The three algebraic integrals are:

$$2(p^2 + q^2) + r^2 = 2c_0x + 6l_1$$
$$2(p\gamma + q\gamma') + r\gamma'' = 2l$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1$$

where l and l_1 are constants of integration. Kovalevsky then derived a fourth integral:

$$\left[(p + qi)^2 + c_0(\gamma + i\gamma')\right]\left[(p - qi)^2 + c_0(\gamma - \gamma' i)\right] = k^2$$

where k is an arbitrary constant. Defining $x_1 = p + qi$, $x_2 = p - qi$, she made several transformations of the variables. After some algebraic manipulations she obtained the equations:

$$0 = \frac{ds_1}{\sqrt{R_1(s_1)}} + \frac{ds_2}{\sqrt{R_1(s_2)}}$$
$$dt = \frac{s_1 ds_1}{\sqrt{R_1(s_1)}} + \frac{s_2 ds_2}{\sqrt{R_1(s_2)}},$$

where $R_1(S)$ is a fifth-degree polynomial whose zeros are unique and s_1 and s_2 are polynomials in x_1 and x_2. This system results in hyperelliptic integrals which Kovalevsky solved by using theta functions.

For this highly praised research Kovalevsky was awarded the Bordin Prize in 1888 and a prize from the Swedish Academy in 1889. Historians, however, have found this work "not of sufficient interest for the theory of a top to find a place here."[43] Today the value of her work is seen not in the results or the originality of the method but in the interest it stimulated in the problem of the rotation of a rigid body. Several researchers have continued the work of finding new cases of special solutions. This includes Chaplygin's development of the integral for a symmetrical top turning about a fixed point, with moments of inertia $A = B = 4C$ and the Hesse-Schiff equations of motion for a top. There is also a body of work by others, for example, Bobylev, Steklov, Goryachev, and V. Kovalevsky.

The remaining works by Sofya Kovalevsky are of lesser importance and will be only briefly reviewed.

In the paper on Abelian integrals, Kovalevsky showed how a certain type of Abelian integral could be expressed as an elliptic integral.

The paper on Saturn's rings is concerned with the stability of motion of liquid ring-shaped bodies. Laplace found the form of the ring to be a skewed cross-section of an ellipse. Kovalevsky, using a series expansion, proved that the rings were egg-shaped ovals symmetric about a single axis. However, the subsequent proof that Saturn's rings consist of discrete particles and not a continuous liquid made this work inapplicable.

In the articles on the refraction of light in crystals, Kovalevsky applied a method developed by Weierstrass to differentiate Lamé's partial differential equations. However, Volterra discovered a basic error in her work. She had (as had Lamé) treated a multi-valued function as though it were single-valued, and the solution could not be applied to the equations in her form. However, the paper did demonstrate the previously unpublished theory of Weierstrass.

Kovalevsky's last article derived a simpler proof of Bruns's Theorem based on a property of the potential function of a homogeneous body.

Notes

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3. *Ibid.*
4. Although Kovalevskaya, the feminine version of the name Kovalevsky, is used in much of the current literature, the author will refer to Kovalevsky in this paper, as that was the name under which Sofya Kovalevsky worked and published. However, in the notes the author has used the version of the name that appeared in the reference.
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22. Kovalevskaya, *Vospominaniya i pis'ma*, p. 352. Cited in G. J. Tee, "Sof'ya Vasil'yevna Kovalevskaya," *Mathematical Chronicle*, 5 (1977) 132–133.

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THE UNIFORMIZATION THEOREM

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Almost one hundred years have passed since Felix Klein discovered the uniformization theorem. While such theorems had earlier been proved in specific cases, no one had dared even conjecture that every compact Riemann surface could be parametrized by a variable whose

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